Moment of inertia

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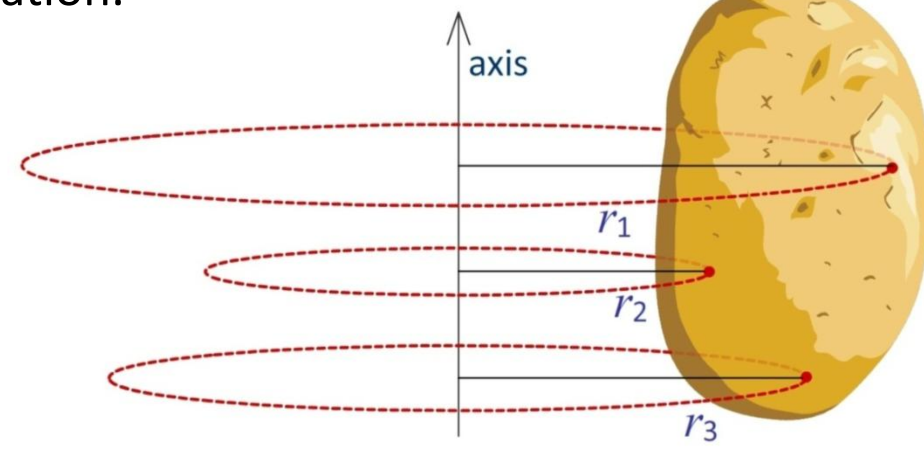
Moment of inertia (I)

It is defined as the sum of product of mass of each particle and square of its perpendicular distance from the axis of rotation.

$$I = \sum mr^2$$

$$I = \int_i^f \mathrm{d}m\, r^2$$

SI unit : kg m^2



Rotational kinetic energy of the system is given by

$$KE = \frac{1}{2}m_1v_1^{\ 2} + \frac{1}{2}m_2v_2^{\ 2} + \frac{1}{2}m_3v_3^{\ 2}\dots$$

Using $v = r\omega$ (ω is constant)

$$KE = \frac{1}{2}m_1r_1^{\ 2}\omega^2 + \frac{1}{2}m_2r_2^{\ 2}\omega^2 + \frac{1}{2}m_3r_3^{\ 2}\omega^2\dots$$

$$KE = \frac{1}{2}\left[m_1r_1^{\ 2} + m_2r_2^{\ 2} + m_3r_3^{\ 2}\dots\right]\omega^2$$

Representing the quantity in the square brackets I

$$KE = \frac{1}{2}I\omega^2$$

Moment of inertia of a thin rod about an axis passing through one of its ends & perpendicular to its length

Let mass of the rod be M and its length be L.
Consider a small segment of mass $\mathrm{d}m$ at a distance x from the axis of rotation.

Moment of inertia due to this segment is

$$\mathrm{d}I = \mathrm{d}m\, x^2$$

$$\mathrm{d}I = \frac{M}{L}\mathrm{d}x\, x^2$$

Total moment of inertia is given by

$$\boxed{I = \int_i^f \mathrm{d}m\, r^2}$$

$$I = \int_0^L \frac{M}{L} x^2\, \mathrm{d}x$$

$$I = \frac{M}{L}\int_0^L x^2\, \mathrm{d}x$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L$$

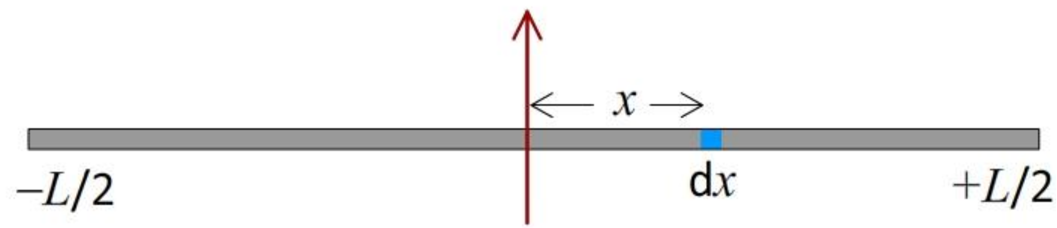
$$I = \frac{M}{L}\left[\frac{L^3}{3} - 0\right]$$

$$I = \frac{M}{L}\left[\frac{L^3}{3}\right]$$

$$I = \frac{ML^2}{3}$$

Moment of inertia of a thin rod about an axis passing through its centre and perpendicular to its length

Let mass of the rod be M and its length be L.
Consider a small segment of mass dm at a distance x from the axis of rotation.



Moment of inertia due to this segment is

$$\mathrm{d}I = \mathrm{d}m\, x^2$$

$$\mathrm{d}I = \frac{M}{L}\mathrm{d}x\, x^2$$

Total moment of inertia is given by

$$\boxed{I = \int_i^f \mathrm{d}m\, r^2}$$

$$I = \int_{-L/2}^{+L/2} \frac{M}{L} x^2\, \mathrm{d}x$$

$$I = \frac{M}{L} \int_{-L/2}^{+L/2} x^2\, \mathrm{d}x$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{+L/2}$$

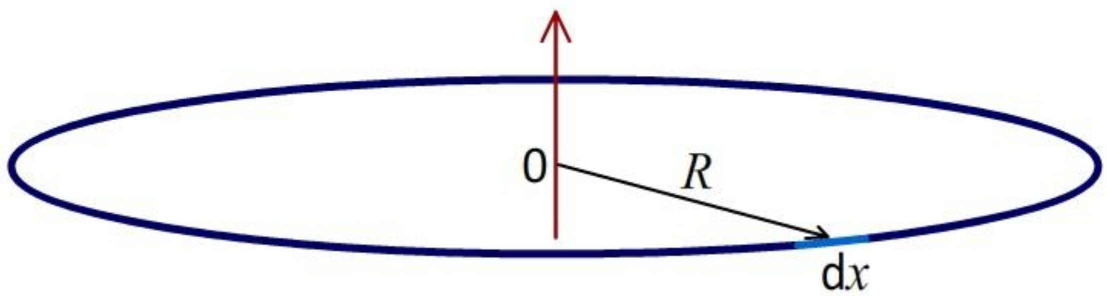
$$I = \frac{M}{3L}\left[\frac{L^3}{8} + \frac{L^3}{8}\right]$$

$$I = \frac{M}{3L}\left[\frac{L^3}{4}\right]$$

$$I = \frac{ML^2}{12}$$

Moment of inertia of a thin ring about an axis passing through its centre and perpendicular to its plane

Let mass of the ring be M and its radius be R.
Consider a small segment of length $\mathrm{d}x$. Let the mass of the segment be $\mathrm{d}m$.



Moment of inertia due to this segment is

$$\mathrm{d}I = \mathrm{d}m\, R^2$$

$$\mathrm{d}I = \frac{M}{2\pi R}\mathrm{d}x\, R^2$$

$$\mathrm{d}I = \frac{MR}{2\pi}\mathrm{d}x$$

Total moment of inertia is given by

$$\boxed{I = \int_i^f \mathrm{d}m\, r^2}$$

$$I = \int_0^{2\pi R} \frac{MR}{2\pi}\, \mathrm{d}x$$

$$I = \frac{MR}{2\pi}\int_0^{2\pi R} \mathrm{d}x$$

$$I = \frac{MR}{2\pi}\left[x\right]_0^{2\pi R}$$

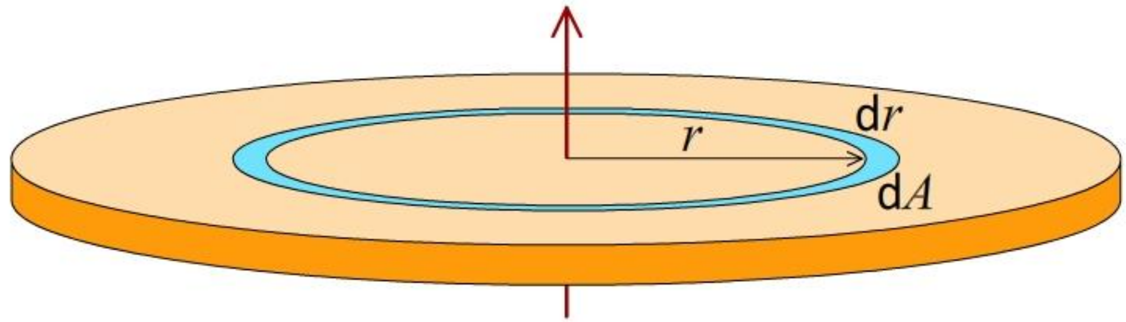
$$I = \frac{MR}{2\pi}\left[2\pi R\right]$$

$$I = MR^2$$

Moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane

Let mass of the disc be M and its radius be R and thickness t

Consider a small segment of radius r and thickness dr.



Moment of inertia due to this segment is

$$\mathrm{d}I = \mathrm{d}m\, R^2$$

$$\mathrm{d}m = \frac{M}{V} \times \mathrm{d}V$$

$$\mathrm{d}I = \frac{M}{\pi R^2 t} 2\pi r\, \mathrm{d}r\, t\, r^2$$

$$\mathrm{d}I = \frac{M}{R^2} 2r\, \mathrm{d}r\, r^2$$

Total moment of inertia is obtained by integration

$$\int \mathrm{d}I = \int \frac{M}{R^2} 2r\, \mathrm{d}r\, r^2$$

$$I = \frac{2M}{R^2} \int r^3\, \mathrm{d}r$$

$$I = \frac{2M}{R^2} \left[\frac{r^4}{4} \right]_0^R$$

$$I = \frac{2M}{R^2} \left[\frac{R^4}{4} - \frac{0^4}{4} \right]$$

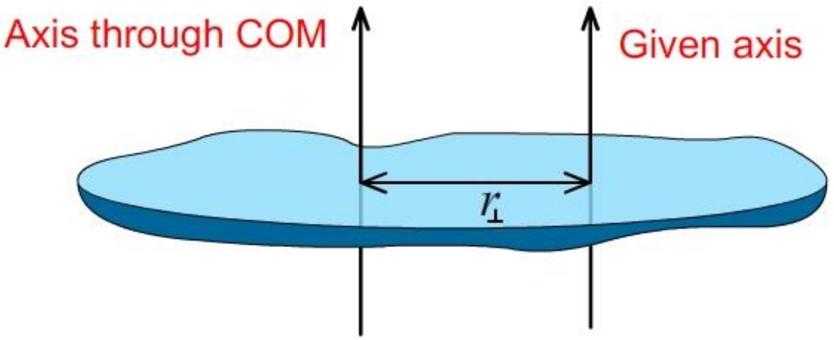
$$I = \frac{2M}{R^2} \left[\frac{R^4}{4} \right]$$

$$I = \frac{MR^2}{2}$$

Parallel axes theorem

Moment of inertial of a body about a given axis is equal to the sum of its moment of inertia about an axis passing through a parallel axis passing through its centre of mass and the product of mass of the body and square of the perpendicular distance between the two axis.

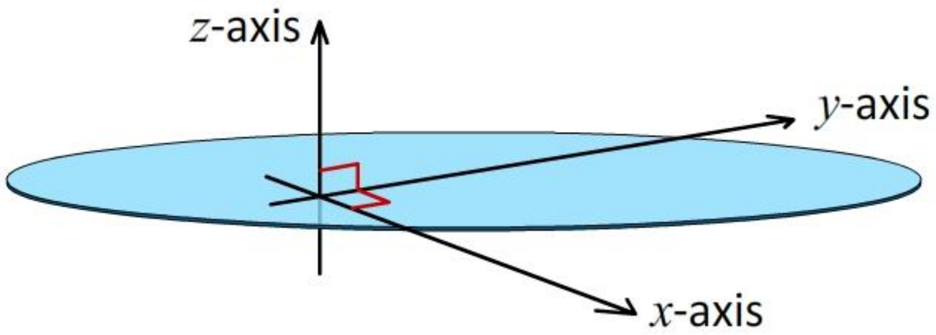
$$I_{\text{given}} = I_{\text{COM}} + MR^2$$



Perpendicular axes theorem

Moment of inertial of a plane laminar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two mutually perpendicular axes in its plane with the perpendicular axis passing through their point of their intersection.

$$I_{\text{Z}} = I_{\text{X}} + I_{\text{Y}}$$



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